Radio-Engineering Means of Transmission, Reception and Processing of Signals

Formation of Minimax Ensembles of Aperiodic Gold Codes

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Abstract

Introduction. Signals constructed on the basis of ensembles of code sequences are widely used in digital communication systems. The development of such systems should be based on the analysis, synthesis and implementation of periodic signal ensembles. Although a number of theoretical methods have been developed for synthesizing periodic signal ensembles, there is a lack of approaches to constructing aperiodic signal ensembles.

Aim. To construct aperiodic Gold code ensembles characterized by optimal values in terms of the ratio of their code length to volume among the currently known binary codes.

Materials and methods. The methods of directed enumeration and discrete choice of the best ensemble based on unconditional preference criteria were used.

Results. Full and truncated aperiodic Gold code ensembles with a given length and ensemble volume were constructed. The parameters and shape of auto- and mutual correlation functions were shown for a number of constructed ensembles. The obtained results were compared with those described in literature for periodic Gold code ensembles regarding an increase in the values of the minimax correlation function depending on the code length and ensemble volume.

Conclusion. The developed algorithms can be used to obtain both full ensembles and ensembles truncated in volume. In addition, the developed algorithms can be extended to the tasks of forming ensembles from other families, e.g., assembled from code sequences belonging to different families.

Keywords aperiodic Gold code sequences, minimax ensembles

Радиотехнические средства передачи, приема и обработки сигналов

Оригинальная статья

Построение минимаксных ансамблей апериодических кодов Голда

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Аннотация

Введение. В системах цифровой связи широко применяются сигналы, построенные на основе ансамблей кодовых последовательностей. При разработке этих систем наибольшее внимание уделяется анализу, синтезу и реализации ансамблей периодических сигналов. Разработаны и используются теоретические методики синтеза ансамблей периодических сигналов. Значительно меньше результатов получено в области построения ансамблей апериодических сигналов с заданными корреляционными свойствами. Теоретические методики синтеза таких ансамблей сигналов практически отсутствуют.

Цель работы. Построение минимаксных ансамблей апериодических кодов Голда, которые обладают одним из лучших среди известных бинарных кодов соотношением длины кодов и объема ансамбля.

Материалы и методы. Для построения минимаксного ансамбля используются направленный перебор и метод дискретного выбора лучшего ансамбля на основе безусловного критерия предпочтения.

Результаты. В статье описан алгоритм формирования полных и неполных минимаксных ансамблей апериодических кодов Голда с заданными длиной и объемом ансамбля. Приведены параметры и вид авто- и взаимно корреляционных функций для ряда полученных ансамблей. Выполнено сравнение результатов статьи с известными результатами для ансамблей периодических кодов Голда в части роста минимаксных значений корреляционных функций в зависимости от длины кодов и объема ансамблей.

Заключение. Разработанные алгоритмы, в отличие от известных, позволяют конструировать как полные ансамбли, так и ансамбли, учитывающие ограничение их объема. Кроме того, данные алгоритмы могут быть распространены на задачи построения ансамблей из других семейств, например собранных из кодовых последовательностей, принадлежащих различным семействам.

Ключевые слова: апериодические последовательности Голда, минимаксные ансамбли

Introduction. Signals constructed on the basis of ensembles of code sequences (CS) are widely used in discrete data transmission radio systems, as well as in radar and radio navigation systems. A prospective research direction in the development of such systems consists in the selection of ensembles exhibiting good correlation properties [1].

Recently, much research effort has been devoted to the creation of ensembles of polyphase and complementary CS, as well as those with a zero correlation zone [1]. To this end, analytical synthesis methods [2–4], genetic [5] or evolutionary algorithms [6], as well as various modifications of these computational procedures [7] are widely applied. At the same time, complementary CS are characterized by such limitations, as the number of sequences in the set, available sequence lengths and the requirement of power amplifier linearity. The latter also pertains to

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polyphase CS. Therefore, the interest in binary CS is still growing. However, the problem of designing sequence families of the required size, sequence length and aperiodic properties remains to be solved [1].

In contemporary radio systems, both periodic and aperiodic discrete signals are implemented. During their simultaneous transmission in the common frequency band, minimization of mutual interference is of great importance, which is achieved by CS minimax ensembles, i.e. CS ensembles optimal in terms of the minimax criterion [8].

For periodic discrete signals, there exist methods for synthesizing CS minimax ensembles [8] and for estimating values of periodic autocorrelation functions (PACF) and periodic cross-correlation functions (PCCF) achievable by such ensembles. However, for aperiodic CS ensembles, there is not only a lack of a versatile method for synthesizing binary phase-manipulated signals optimal by the minimax criterion, but also it is unclear whether the known signals with a large number of N positions are close to optimal [9].

Moreover, as indicated in [9], despite intensive studies, all existing methods for synthesizing single aperiodic CS include enumeration as one of the stages. Among them are [9, 10]:

1. Directed enumeration method. All the binary Barker sequences, ternary aperiodic quasi-orthogonal CS, including ternary Barker sequences of $N < 31$, were obtained using this method. The minimum peak value of the side lobes for the aperiodic autocorrelation function (AACF) was determined up to $N = 105$ [11].

The directed enumeration method involves two stages. The first stage is focused on narrowing the enumerated area, consisting in formulating the necessary conditions and admissible combinatorial parameter relations. The second stage involves the development of efficient enumeration algorithms.

2. Synthesis of aperiodic CS on the basis of periodic CS. This method is based on the relationship of $\rho_a(m)$ AACF and its $\rho_p(m)$ PACF periodic counterpart. If

$$\rho_{p_{max}} = \max\left\{\left|\rho_p(m)\right|\right\}; \ m = 1, \ 2, \ \ldots, \ N-1,$$

an estimate of $\rho_{a_{max}}(m) \geq (1/2)\rho_{p_{max}}$ can be obtained. Thus, sequences with "good" AACFs can only be found among sequences with "good" PACF [12].

This approach also comprises two stages. The first implies a search for a CS with a "good" PACF. The second implies a search for initial conditions

optimal by the minimax criterion. Following this approach, binary and ternary sequences optimal by the minimax criterion can be obtained.

3. Synthesis of CS signals according to a given AACF. Depending on the criterion used and the method for calculating deviations, the following methods are distinguished [9]:

– uniform approximation method;
– minimum RMSD method;
– coordinate descent method;
– minimum generalized mean deviation method;
– asymptotic synthesis method.

All these methods refer to the group of iterative methods and imply a laborious search process. The best results of $N < 901$ binary sequence synthesis are significantly inferior to the minimax AACF side lobes synthesized by one of the previously mentioned methods.

As noted in [9], the problem of improving these methods consist in the synthesis of CS with "good" PACF, as well as in reducing the duration of the enumeration stage.

Problem statement. The method of constructing aperiodic CS on the basis of periodic CS can also be applied for constructing minimax ensembles of aperiodic CS.

For minimax ensembles of binary periodic CSs, $\rho_{p_{max}}$ estimates are known, depending on the N code length and the K ensemble volume. For a number of popular ensembles, such estimates are provided in [12] (Table 1). As can be seen from Table 1, for large K values close to N, Gold codes are recommended. For example, only Gold codes with a length of $N = 127$ and above and Kasami ensembles with a length of $N = 1023$ and above satisfy the $K_c = 100$ required ensemble volume. This determined the choice of a Gold code ensemble in the present study for subsequent construction of an aperiodic CS minimax ensemble, although, for periodic Gold codes, the $\rho_{p_{max}}$ estimate is somewhat worse than for other ensembles provided in Table 1.

The present article aims to construct minimax ensembles of aperiodic Gold code sequences with a volume close to the code length. It should be noted that Gold ensembles are extremely popular in contemporary code division multiple access (CDMA) systems including, in particular, GPS, UMTS, etc. [13]. Such ensembles are applied for extending sequences converting an information signal into a

Table 1. Characteristics of pseudo-random sequence ensembles

Ensemble	Length, N	Volume, K	Squared maximum ACF sidelobe, $\rho^2_{\text{p}_{\max}}$
Gold	$2^n - 1, \ n \neq 0 \bmod 4;$ 7, 31, 63, 127, 511, 1023	$N + 2 = 2^n + 1$	$\dfrac{\left[\sqrt{2(N+1)}+1\right]^2}{N^2} \to \dfrac{2}{N},$ n is odd; $\dfrac{\left[\sqrt{2(N+1)}+1\right]^2}{N^2} \to \dfrac{4}{N},$ n is even
Kasami	$2^n - 1, \ n$ is even; 15, 63, 255, 1023	$\sqrt{N+1}$	$\dfrac{\left(\sqrt{N+1}+1\right)^2}{N^2} \to \dfrac{1}{N}$
Union of Kasami and bent-sequences	$2^n - 1, \ n = 0 \bmod 4;$ 15, 255	$2\sqrt{N+1} - 1$	$\dfrac{\left(\sqrt{N+1}+1\right)^2}{N^2} \to \dfrac{1}{N}$
Kamaletdinov 1	$p(p-1), \ p$ is prime; 42, 110, 343, 506, 930	$p + 1 = \dfrac{\sqrt{4N+1}+3}{2} \to \sqrt{N}$	$\dfrac{(p+3)^2}{N^2} \to \dfrac{1}{N}$
Kamaletdinov 2	$p(p+1), \ p$ is prime; 12, 56, 132, 380, 552, 930	$p - 1 = \dfrac{\sqrt{4N+1}-3}{2} \to \sqrt{N}$	$\dfrac{(p+1)^2}{N^2} \to \dfrac{1}{N}$

broadband one, as well as for separating data transmitted over the communication line between different subscribers and synchronizing the reception of information packets.

Let $\mathbf{a}_{ki} = \{a_{ki,0}, \ a_{ki,1}, \ \dots, \ a_{ki,N-1}\}$ be the k-th binary aperiodic CS with the N length, belonging to the i-th Gold ensemble of $\{\mathbf{a}_k\}_i$, $k = 1, \ \dots, \ K$; $i = 1, \ \dots, \ N_{\text{Gold}}$. The normalized AACF of the $\mathbf{a}_{ki}$ sequence is defined as [12]:

$$\rho_{\text{a}, \, kki}(m) = \begin{cases} \dfrac{1}{\|\mathbf{a}_{ki}\|^2} \displaystyle\sum_{j=m}^{N-1} a_{ki,j} a_{ki,j-m}, \ m \geq 0; \\ \dfrac{1}{\|\mathbf{a}_{ki}\|^2} \displaystyle\sum_{j=0}^{N-1+m} a_{ki,j} a_{ki,j-m}, \ m < 0, \end{cases}$$

where $\|\mathbf{a}_{ki}\|$ is the Euclidean norm, the same for all code vectors of $\mathbf{a}_{ki}$; $\|\mathbf{a}_{ki}\|^2 = E$ is the energy of each $\mathbf{a}_{ki}$ code sequence.

The normalized ACCF for $\mathbf{a}_{ki}$ and $\mathbf{a}_{li}$ two sequences of the same length is

$$\rho_{\text{a}, \, kli}(m) = \begin{cases} \dfrac{1}{\|\mathbf{a}_{ki}\| \|\mathbf{a}_{li}\|} \displaystyle\sum_{j=m}^{N-1} a_{ki,j} a_{li,j-m}, \ m \geq 0; \\ \dfrac{1}{\|\mathbf{a}_{ki}\| \|\mathbf{a}_{li}\|} \displaystyle\sum_{j=0}^{N-1+m} a_{ki,j} a_{li,j-m}, \ m < 0. \end{cases}$$

Let

$$R_{kk, \, i} = \max_m \left| \rho_{\text{a}, \, ki}(m) \right|, \ m \neq 0$$

be the side lobe peak level of k-th AACF for the i-th ensemble of the aperiodic Gold CS;

$$Q_{kl, \, i} = \max_m \left| \rho_{\text{a}, \, kli}(m) \right|, \ k \neq l$$

is the maximum value of the ACCF module for the $\mathbf{a}_{ki}$ and $\mathbf{a}_{li}$ sequences. For the i-th ensemble, let us find a pair of values: $\max_k R_{kk, \, i}$ achieved with some CS of $k = j_1$ and $\max_{k,l} Q_{kl, \, i}$ achieved with some CS pair of $k = j_2$, $l = j_3$. With the K and N given values, in terms of the $\{\mathbf{a}_{kM}\}$, $k = 1, \ \dots, \ K$; $m = 0, \ \dots, \ N-1$ minimax ensemble, let us name the ensemble chosen as follows:

1) for each i-th ensemble, a pair of $\max_k R_{kk, \, i}$ and $\max_{k,l} Q_{kl, \, i}$ values is determined;

2) on the basis of the two-criteria choice algorithm [17], the $i = M$ ensemble is obtained having the best pair of $\max_k R_{kk, \, i}$ and $\max_{k,l} Q_{kl, M}$ values according to the specified algorithm.

According to Table 1, the full volume of the Gold ensemble is equal to $K = N + 2$.

This article discusses both full and truncated minimax ensembles of $K_c < K$ volume, which are necessary for a number of practical applications. The truncated minimax ensembles included in the analysis are formed from the full Gold minimax ensemble with the $i = M$ number by truncating $K - K_c$ sequences with the highest values of $R_{kk, M}$.

Algorithm for constructing a minimax ensemble of aperiodic Gold codes. On the basis of the definition provided above, this section describes a procedure for constructing a minimax ensemble, which consists of the following operations:

– formation of Gold code ensembles;

– choice of the best Gold ensemble from the set on the basis of the two-criteria choice algorithm [17].

According to the Gold method [14], a Gold code ensemble is formed by selecting pairs of m-sequences based on the properties of polynomials. Each m-sequence of the $N = 2n - 1$ length (n is an integer) is corresponded to its own primitive polynomial of n degree. Primitive polynomials are usually given in tables [15] with their number being equal to $P = \Phi\left(2^n - 1\right)\big/n$, where $\Phi(x)$ is the Euler function (the number of natural numbers less than and mutually prime with x). The number of mismatching pairs for m-sequences constructed from primitive polynomials is equal to $N_a = \left(P^2 - P\right)\big/2$.

Gold ensembles cannot be obtained for all combinations of m-sequences of a certain length. For generating Gold codes, preferred pairs of m-sequences are selected on the basis of the following algorithm:

– all the n-degree primitive polynomials forming m-sequences are obtained;

– each of the obtained m-sequences is decimated by the q decimation coefficients presented below;

– from the m-sequences obtained after decimation, the polynomials generating them are obtained using the Berlekamp–Massey algorithm [16];

– mirror copies of the obtained polynomial pairs are discarded.

The remaining pairs represent the preferred pairs.

In order to obtain the decimation coefficient, the following relationship between the roots of some primitive polynomials is used: the roots of the one $f_r(x)$ polynomial are the q-th degree roots of another polynomial $f_l(x)$ with the q number being coprime with N. Then the M_l m-sequence generated by the $f_l(x)$ polynomial can be formed by selecting each q-th element from the M_r m-sequence.

The q decimation coefficients lead to the formation of preferred pairs, provided the following conditions are met: $q = 2^k + 1$ or $q = 2^{2k} - 2^k + 1$, where $k \le (n-1)/2$ and the greatest common divisor of k and n numbers is equal to GCD(k, n) = 1.

When determining pairs of m-sequences for constructing a Gold ensemble, ensembles based on cyclic shifts of the original m-sequences were not considered. Enumeration of the shifted CS can serve as an additional reserve for optimization.

Let us consider the problem of compiling preferred m-sequence pairs using the example of a $N = 127$ length sequence generated by a primitive polynomial of the $n = 7$ degree. For $n = 7$, 18 primitive polynomials are available, 9 of which are provided in [15]. The other nine represent their mirror polynomials. All 18 polynomials correspond to mutually inverse m-sequences, i.e. m-sequence pairs, connected by a decimation coefficient:

$$q = \left[2^j (N-1)\big/2\right] \bmod N.$$

Thus, by the n degree of the primitive polynomial, all possible ensembles of Gold codes are determined. This article restricts to $n = 5$, 6, 7, 9 and 10. The calculated parameters for these n values are given in Table 2.

Subsequently, for the received Gold code ensembles, the $\max\limits_{k} R_{kk,i}$ and $\max\limits_{k,l} Q_{kl,i}$ values are determined by enumeration.

The choice of the best Gold ensemble in terms of the minimax criterion represents the task of a two-criterion discrete choice. To this end, the $\left\{\max\limits_{k} R_{kk,i};\ \max\limits_{k,l} Q_{kl,i}\right\}$ resulting set is divided into sets of "worst" and "not worst" cases by applying an unconditional preference criterion. Next, the rectangle method [17] is applied illustrated in Fig. 1 for $N = 127$ and 511. The rectangle method consists of the following:

1. The $\max\limits_{k} R_{kk,i}$ and $\max\limits_{k,l} Q_{kl,i}$ indicators are plotted along the axes of the coordinate plane in ascending order.

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Table 2. Number of Gold ensembles

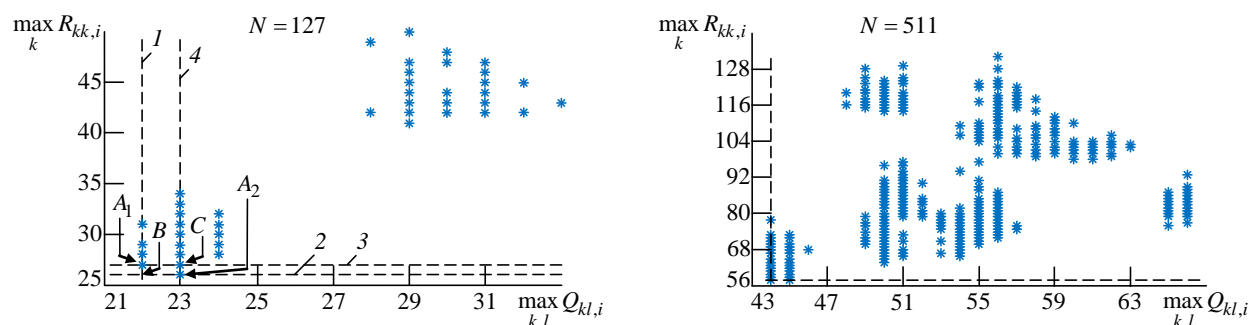| Power of generating polynomial n	Sequence length N	Number of primitive polynomials P	Number of m-sequence pairs N_{a}	Number of Gold ensembles N_{Gold}
5	31	6	15	12
6	63	6	15	6
7	127	18	153	90
9	511	48	1128	288
10	1023	60	1770	300

Fig. 1. Distribution of maximum values of ensemble correlation functions for two length values

2. On this plane, the points of the $\left\{ \max_{k} R_{kk,i};\ \max_{k,l} Q_{kl,i} \right\}$ set are plotted.

3. On the vertical line *1* passing through the leftmost point of the set, either the A_1 lowest or the only point is selected.

4. The horizontal line *2* is drawn through the lowest point of the set. When this line contains several points, the (A_2) leftmost point is selected. The A_1, A_2 points obtained in this way are the extreme points of the lower left border. The intersection point of the drawn lines is called *B*.

5. The horizontal line *3* and the vertical line *4* are drawn through the A_1 and the A_2 points, respectively, until their intersection at the *C* point. All points outside the resulting rectangle are excluded from further selection.

6. Inside the $A_1 C A_2 B$ rectangle, a vertical line and a horizontal line are drawn through the leftmost point (or points) and the lowest point (or points), respectively. Then, the leftmost point on the A_3 horizontal line and the lowest point on the A_4 vertical line will be the next points of the lower left boundary.

7. Steps 3–6 are repeated as many times as it is possible to draw new lines. Thus, a point of the set with the minimum *i*-th values for $\max_{k} R_{kk,i}$ and $\max_{k,l} Q_{kl,i}$ is obtained. If several ensembles with the same values of indicators are obtained, these ensembles are considered identical in their characteristics, thus being fit for selection.

Results. In accordance with the described algorithms, full and truncated Gold ensembles were constructed for the parameters indicated in Table 2. The selection of optimal minimax ensembles was performed. In Table 3, conventional numbers of preferred pairs for primitive polynomials are given, resulting in minimax $(K_{\mathrm{c}} = K = N + 2)$ full and $(K_{\mathrm{c}} = 100)$ truncated ensembles, as well as the values of the AACF and ACCF side lobes.

As can be seen in Fig. 1, for $N = 127$ and $N = 511$, two and one best optimization results are determined, respectively, for the distribution of $\max_{k} R_{kk,i}$ and $\max_{k,l} Q_{kl,i}$. This fact is reflected in Table 3, where, for $n = 7$, two sets of minimax values are given.

Fig. 2 demonstrates the superimposed AACF of minimax full and truncated $(K_{\mathrm{c}} = 100)$ ensembles from Table 3 for $N = 127$ with analogous AACF for $N = 511$ represented in Fig. 3.

Fig. 4 shows the superimposed ACCF of the Gold code ensemble selected by the algorithm [17]. Fig. 5 demonstrates the dependencies of the $\min_{i} \max_{k} R_{kk,i}$ AACF minimax level on the volume of the K_{c} ensemble for codes of the N various length.

Table 3. Numbers of preferred pairs of primitive polynomials that yield minimax ensembles

Power of generating polynomial, n	Sequence length, N	Ensemble volume	Numbers of first primitive polynomials in a pair	Numbers of second primitive polynomials in a pair	AACF sidelobe minimax value, $\min_i \max_k R_{kk,\,i}$	ACCF minimax value, $\min_i \max_{k,\,l} Q_{kl,\,i}$
5	31	$N+2$	2	6	11	13
			2	3	12	12
			2	5	12	12
			3	4	12	12
			4	6	12	12
6	63	$N+2$	2	5	19	19
7	127	100	5	6	22	27
			14	18	23	26
		$N+2$	3	15	25	28
			14	18	26	26
9	511	100	3	17	44	58
		$N+2$	3	17	57	58
10	1023	100	45	49	65	91
			33	53	64	89
		$N+2$	33	53	89	89

Discussion. The application of the widely-used method for constructing Gold ensembles followed by selecting the best ensemble on the basis of the unconditional preference criterion [17] allowed minimax ensembles of aperiodic Gold codes to be constructed. The developed algorithms ensure the construction of both full and limited in volume (truncated) ensembles. In addition, these algorithms can be extended to the problem of constructing ensembles from other families, e.g., assembled from CS belonging to different families.

Let us compare the obtained values characterizing minimax ensembles of aperiodic Gold CS with those published in literature.

1. There exist fundamental restrictions on the side lobes of AACF. For arbitrary single binary CS, the

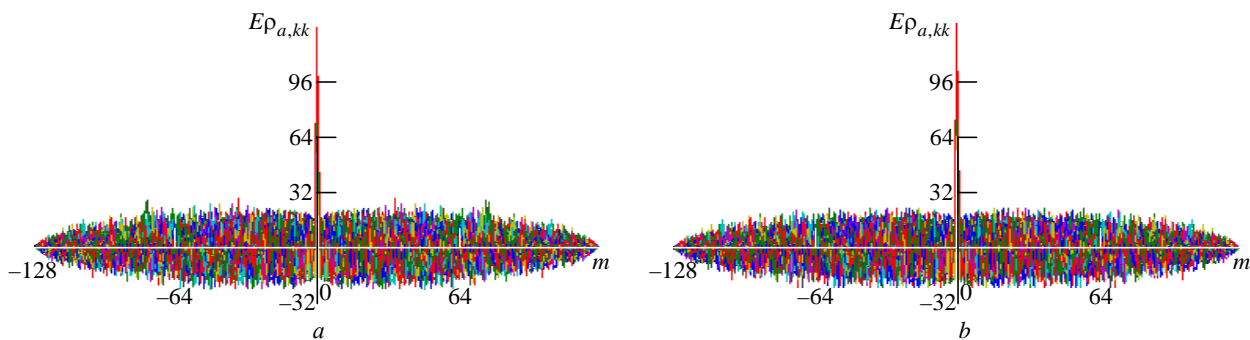$$SL_{\max} = \max_m |\rho_{a,\,kki}(m)|, \ m \neq 0$$

Fig. 2. Gold codes length 127 aperiodic autocorrelation functions: a – full ensemble; b – truncated ensemble $K_c = 100$
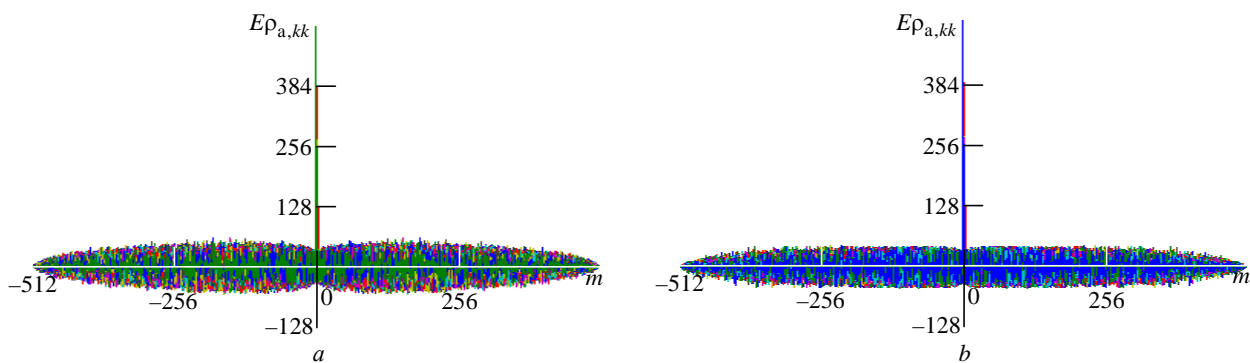

Fig. 3. Gold codes length 511 aperiodic autocorrelation functions: a – full ensemble; b – truncated ensemble $K_c = 100$
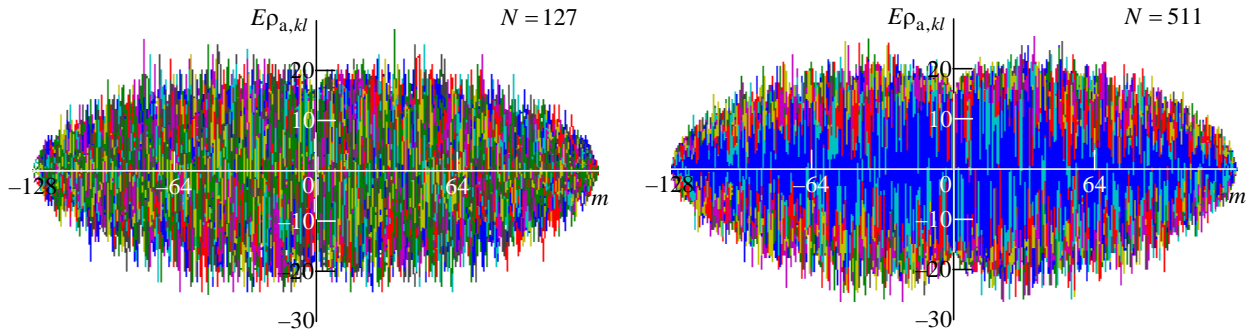
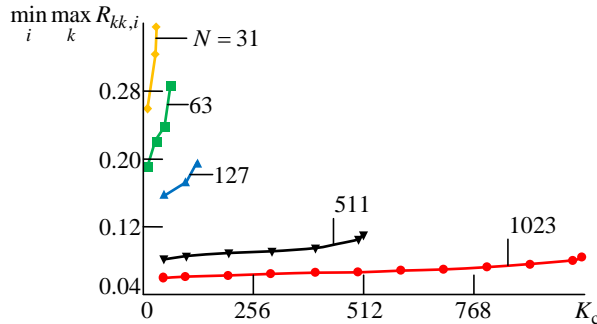*Fig. 4.* Aperiodic cross-correlation functions of selected Gold codes

Fig. 5. The maximum level of the side lobe of the aperiodic autocorrelation function versus the volume of the ensemble

module of the maximum side lobe of the normalized AACF is bounded from below by $1/N$. It is known that the lower boundary is determined by Barker codes. For non-Barker codes, possible values comprise $SL_{\max} \geq 2/N$.

In previous research, authors established minimum peak values of the AACF side lobes

$$SL_{\min} = \min SL_{\max}$$

and N corresponding ranges among all binary CSs (Table 4 [11]).

According to Fig. 5, for small code lengths $(N = 31, 63, 127)$, the minimax values for the AACF lobes of the full Gold ensembles obtained in this article significantly exceed those specified in Table 4. However, as compared to K, a decrease in the volume of ensembles to K_c leads to a sharp decrease in minimax lobes.

Table 4. Minimum sidelobe levels of pseudo random sequence auto-correlation functions

Sequence length, N	Sidelobe minimum, $SL_{\min}$	Normalized sidelobe minimum, $SL_{\min}/N$
5	1	0.2
6…21 (except Barker codes)	2	0.333…0.095
22…48	3	0.136…0.063
49…82	4	0.081…0.048
83…105	5	0.060…0.048

Due to the absence of available literature data, it was impossible to compare the obtained minimax values for the side lobes of full ensembles with the absolutely minimal normalized values of $SL_{\min}/N$ for $N = 511$ and 1023. However, it should be noted that, for these lengths, the values of the minimax lobes decrease significantly with the limited ensemble length in comparison with $N = 31$, 63, 127. In addition, for insignificant lengths, a decrease in the ensemble volume, in comparison with the full one, provides no significant decrease in minimax lobes. At the same time, when obtaining, e.g., a $K_c = 100$ ensemble, then, for $N = 511$, 1023, the values of minimax lobes are close to those indicated in Table 4 for $N = 100$.

2. In [8], periodic CS ensembles minimax by the criterion are considered:

$$P_{\mathrm{opt}} = \min \max \left[\left| c_{k,\,l}(m) \right|_{k \neq l} \cup \left| c_{k,\,k}(m) \right|_{m \neq 0} \right], \quad (1)$$

where $c_{k,\,l}(m)$ is the PCCF of the k and l sequences; $c_{k,\,k}(m)$ is the PACF of the k-th sequence.

For $K > N$ volumes and odd n, the Grz lower boundaries for periodic binary Gold CSs satisfying (1) coincide with the values of $\rho_{\mathrm{p_{max}}}$ obtained from Table 1, i.e. these ensembles are strictly optimal by the minimax criterion (1).

Table 5 presents the Grz and $\rho_{\mathrm{p_{max}}}$ values for minimax ensembles of periodic Gold CS and the obtained values of

$$\rho_{\mathrm{a_{max}}} = \max \left[\min_{i} \max_{k} R_{kk,\,i},\ \min_{i} \max_{k,\,l} Q_{kl,\,i} \right].$$ According to Table 5, the minimax values for aperiodic Gold CSs are 1.5...2 times higher than the corresponding values for periodic Gold CS.

3. A number of works (e.g., [4]) discuss the possibility of changing the maximum level of side lobes

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*Table 5.*Comparison of auto-correlation function side lobe levels

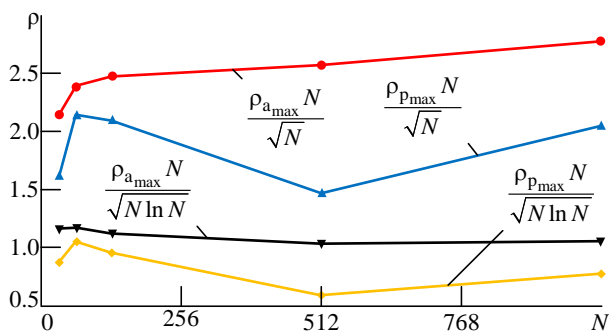| Power of generating polynomial, n	Sequence length, N	Sidelobe level lower bound, $\rho_{p,\,гр}$	Maximum PACF sidelobe level, $\rho_{p_{max}}$	Maximum AACF sidelobe level, $\rho_{a_{max}}$
5	31	0.29	0.29	0.387
6	63	0.206	0.27	0.302
7	127	0.186	0.186	0.22
9	511	0.065	0.065	0.114
10	1023	0.046	0.064	0.087

Fig. 6. Auto-correlation function side lobes of Gold ensembles

for various CS ensembles, compared with the $\sqrt{N}$ and $\sqrt{N \ln N}$ functions. Fig. 6 represents the N-dependent values of the lobe minimax levels for the periodic and aperiodic Gold ensembles normalized to the indicated values. The nature of these dependenc-es illustrates that the $\rho_{p_{max}} N$ and $\rho_{a_{max}} N$ grow approximately in proportion to $\sqrt{N}$, . This agrees well with the results obtained in [11] for the maximum levels of side lobes of periodic Gold CS ensembles. In order to confirm this for $N > 1023$, further calculations are required.

4. For truncated minimax Gold CS ensembles, it is possible to obtain $\rho_{a_{max}} < \rho_{p_{max}}$ by increasing the n degree of the polynomial by one under the equal volume of K_c ensembles. Thus, for a $K_c = 65$ periodic ensemble, the values of $\rho_{p_{max}} = 0.27$ and $\rho_{a_{max}} = 0.17$ are obtained for the $N = 63$ and $N = 127$, respectively.

Author's contribution

Boris V. Ponomarenko – problem statement, selection and rationale of approaches to the problem solving.
Aleksey G. Vostretsov – analysis of the results obtained by comparing with foreign information sources.
Vladimir Yu. Zubarev, Evgeniy G. Shanin – development and modeling of algorithms.

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Пономаренко Б. В. – постановка задачи, выбор и обоснование подходов к ее решению.
Вострецов А. Г. – сравнительный анализ полученных результатов с зарубежными источниками.
Зубарев В. Ю., Шанин Е. Г. – разработка и моделирование алгоритмов.

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